

Ex 1

List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

- a) $a = b$.
- b) $a + b = 4$.
- c) $a > b$.
- d) $a \mid b$.
- e) $\gcd(a, b) = 1$.
- f) $\text{lcm}(a, b) = 2$.

Ex 2

- a) List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$.
- b) Display this relation graphically, as was done in Example 4.
- c) Display this relation in tabular form, as was done in Example 4.

Ex 3

Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

- a) $x \neq y$.
- b) $xy \geq 1$.
- c) $x = y + 1$ or $x = y - 1$.
- d) $x \equiv y \pmod{7}$.
- e) x is a multiple of y .
- f) x and y are both negative or both nonnegative.
- g) $x = y^2$.
- h) $x \geq y^2$.

Ex 4

 Let R be a relation from a set A to a set B . The **inverse relation** from B to A , denoted by R^{-1} , is the set of ordered pairs $\{(b, a) \mid (a, b) \in R\}$. The **complementary relation** $\bar{R}$ is the set of ordered pairs $\{(a, b) \mid (a, b) \notin R\}$.

Let R be the relation $R = \{(a, b) \mid a < b\}$ on the set of integers. Find

- a) R^{-1} .
- b) $\bar{R}$.

Ex 5

- $R_1 = \{(a, b) \in \mathbf{R}^2 \mid a > b\}$, the “greater than” relation,
- $R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \geq b\}$, the “greater than or equal to” relation,
- $R_3 = \{(a, b) \in \mathbf{R}^2 \mid a < b\}$, the “less than” relation,
- $R_4 = \{(a, b) \in \mathbf{R}^2 \mid a \leq b\}$, the “less than or equal to” relation,
- $R_5 = \{(a, b) \in \mathbf{R}^2 \mid a = b\}$, the “equal to” relation,
- $R_6 = \{(a, b) \in \mathbf{R}^2 \mid a \neq b\}$, the “unequal to” relation.

Find

- | | |
|------------------------------|------------------------------|
| a) $R_1 \cup R_3$. | b) $R_1 \cup R_5$. |
| c) $R_2 \cap R_4$. | d) $R_3 \cap R_5$. |
| e) $R_1 - R_2$. | f) $R_2 - R_1$. |
| g) $R_1 \oplus R_3$. | h) $R_2 \oplus R_4$. |

Find

- | | |
|-----------------------------|-----------------------------|
| a) $R_1 \circ R_1$. | b) $R_1 \circ R_2$. |
| c) $R_1 \circ R_3$. | d) $R_1 \circ R_4$. |
| e) $R_1 \circ R_5$. | f) $R_1 \circ R_6$. |
| g) $R_2 \circ R_3$. | h) $R_3 \circ R_3$. |

Ex 6

Let R_1 and R_2 be the “divides” and “is a multiple of” relations on the set of all positive integers, respectively. That is, $R_1 = \{(a, b) \mid a \text{ divides } b\}$ and $R_2 = \{(a, b) \mid a \text{ is a multiple of } b\}$. Find

- | | |
|------------------------------|----------------------------|
| a) $R_1 \cup R_2$. | b) $R_1 \cap R_2$. |
| c) $R_1 - R_2$. | d) $R_2 - R_1$. |
| e) $R_1 \oplus R_2$. | |

Ex 7

Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 3)$, $(2, 4)$, $(3, 1)$, $(3, 4)$, $(3, 5)$, $(4, 2)$, $(4, 5)$, $(5, 1)$, $(5, 2)$, and $(5, 4)$. Find

- a)** R^2 . **b)** R^3 . **c)** R^4 . **d)** R^5 .

Ex 8

Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrix representing

- a)** R^{-1} . **b)** $\overline{R}$. **c)** R^2 .

Ex 9

Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the matrices that represent

- | | | |
|-----------------------------|------------------------------|-----------------------------|
| a) $R_1 \cup R_2$. | b) $R_1 \cap R_2$. | c) $R_2 \circ R_1$. |
| d) $R_1 \circ R_1$. | e) $R_1 \oplus R_2$. | |

Ex 10

Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find the matrices that represent

- a)** R^2 . **b)** R^3 . **c)** R^4 .

Ex 11

Consider the equivalence relation from Example 2, namely, $R = \{(x, y) \mid x - y \text{ is an integer}\}$.

- a)** What is the equivalence class of 1 for this equivalence relation?
b) What is the equivalence class of $1/2$ for this equivalence relation?

Solution 1

(a) $R = \{(0,0), (1,1), (2,2), (3,3)\}$

(b) $R = \{(1,3), (2,2), (3,1), (4,0)\}$

(c) $R = \{(1,0), (2,0), (3,0), (4,0), (2,1), (3,1), (4,1), (3,2), (4,2), (4,3)\}$

(d) $R = \{(1,0), (2,0), (3,0), (4,0), (1,1), (1,2), (2,3), (1,3), (3,3)\}$

(e) $R = \{(1,0), (0,1), (1,1), (1,2), (1,3), (2,1), (3,1), (4,1), (2,3), (3,2), (4,3)\}$

(f) $R = \{(1,2), (2,1), (2,2)\}$

Given:

$$A = \{0,1,2,3,4\}$$

$$B = \{0,1,2,3\}$$

(a)

$$R = \{(a,b) | a = b\}$$

We note that the elements common by A and B are 0, 1, 2 and 3.

R then contains the points with $a = b = 0$, $a = b = 1$, $a = b = 2$ and $a = b = 3$

$$R = \{(0,0), (1,1), (2,2), (3,3)\}$$

(b)

$$R = \{(a,b) | a + b = 4\}$$

We note that the sum of the element of A and the element of B needs to be equal to 4:

$$a = 1, b = 3$$

$$a = 2, b = 2$$

$$a = 3, b = 1$$

$$a = 4, b = 0$$

R then contains the points:

$$R = \{(1,3), (2,2), (3,1), (4,0)\}$$

(c)

$$R = \{(a,b)|a > b\}$$

We note that the element of A needs to be larger than the element of B :

$$a = 1, b = 0$$

$$a = 2, b = 0$$

$$a = 3, b = 0$$

$$a = 4, b = 0$$

$$a = 2, b = 1$$

$$a = 3, b = 1$$

$$a = 4, b = 1$$

$$a = 3, b = 2$$

$$a = 4, b = 2$$

$$a = 4, b = 3$$

R then contains the points:

$$R = \{(1,0), (2,0), (3,0), (4,0), (2,1), (3,1), (4,1), (3,2), (4,2), (4,3)\}$$

(d)

$$R = \{(a, b) | a|b\}$$

We note that the element of A needs to be a divisor of the element of B .

All nonnegative integers are divisors of 0 (as 0 divided by any nonnegative integer is 0)

$$a = 1, b = 0$$

$$a = 2, b = 0$$

$$a = 3, b = 0$$

$$a = 4, b = 0$$

The only divisor of 1 is 1 itself.

$$a = 1, b = 1$$

The divisors of 2 are 1 and 2.

$$a = 1, b = 2$$

$$a = 2, b = 2$$

The divisors of 3 are 1 and 3

$$a = 1, b = 3$$

$$a = 3, b = 3$$

R then contains the points:

$$R = \{(1, 0), (2, 0), (3, 0), (4, 0), (1, 1), (1, 2), (2, 2), (1, 3), (3, 3)\}$$

(e)

$$R = \{(a,b) | \gcd(a,b) = 1\}$$

We note that the greatest common divisor of the element of A and the element of B needs to be equal to 1. In other words, 1 can be the only common divisor of the two elements.

Since 0 is divisible by any nonnegative integer and since 1 is only divisible by 1, the greatest common divisor of 0 and 1 is 0.

$$a = 1, b = 0$$

$$a = 0, b = 1$$

Since 1 is only divisible by 1, the greatest common divisor between 1 and any positive integer is always 1.

$$a = 1, b = 1$$

$$a = 1, b = 2$$

$$a = 1, b = 3$$

$$a = 2, b = 1$$

$$a = 3, b = 1$$

$$a = 4, b = 1$$

The greatest common divisor of 2 and 3 is 1.

$$a = 2, b = 3$$

$$a = 3, b = 2$$

The greatest common divisor of 2 and 4 is 2.

The greatest common divisor of 3 and 4 is 1.

$$a = 4, b = 3$$

R then contains the points:

$$R = \{(1,0), (0,1), (1,1), (1,2), (1,3), (2,1), (3,1), (4,1), (2,3), (3,2), (4,3)\}$$

(f)

$$R = \{(a, b) | lcm(a, b) = 2\}$$

We note that the least common multiple of the element of A and the element of B needs to be equal to 2.

Since the least common multiple is 2, neither element can be larger than 2.

Since any multiple of 0 is always 0 itself, the least common multiple of 0 and any other integer does not exist.

If both elements are 1, then the least common multiple is 1.

The only remaining possibility is then that at least one of the two elements is a 2 and the other element is a 1 or 2. In each of these cases, the least common multiple is 2.

$$a = 1, b = 2$$

$$a = 2, b = 1$$

$$a = 2, b = 2$$

R then contains the points:

$$R = \{(1, 2), (2, 1), (2, 2)\}$$

Solution 2

A. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),$
 $(2, 2), (2, 4), (2, 6),$
 $(3, 3), (3, 6),$
 $(4, 4),$
 $(5, 5),$
 $(6, 6)\}$



R	1	2	3	4	5	6
1	x	x	x	x	x	x
2		x		x		x
3			x			x
4				x		
5					x	
6						x

Solution 3

DEFINITIONS

A relation R on a set A is **reflexive** if $(a, a) \in R$ for every element $a \in A$.

A relation R on a set A is **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$

A relation R on a set A is **antisymmetric** if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$

A relation R on a set A is **transitive** if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$

SOLUTION

$A =$ Set of all integers

(a)

$$R = \{(x, y) | x \neq y\}$$

The relation R is **not reflexive**, because $x \neq x$ is never true.

The relation R is **symmetric**, because if $x \neq y$, then $y \neq x$.

The relation R is **not antisymmetric**, because $1 \neq 2$ and $2 \neq 1$, while 1 and 2 are not the same.

The relation R is **not transitive**, because $1 \neq 2$ and $2 \neq 1$, while $1 = 1$.

(b)

$$R = \{(x, y) | xy \geq 1\}$$

The relation R is **not reflexive**, because when $x = 0$: $xx = x^2 = 0 < 1$ and thus $(0, 0) \notin R$.

The relation R is **symmetric**, because if $xy \geq 1$, then $yx = xy \geq 1$ (commutative property of multiplication).

The relation R is **not antisymmetric**, because $1 \cdot 2 \geq 1$ and $2 \cdot 1 \geq 1$, while $1 \neq 2$.

The relation R is **transitive**, because if $xy \geq 1$ and if $yz \geq 1$, then x and y are nonzero integers of the same sign, and y and z are nonzero integers of the same sign. Then x and z are nonzero integers of the same sign and thus $xz \geq 1$.

(c)

$$R = \{(x, y) | x = y + 1 \text{ or } x = y - 1\}$$

The relation R is **not reflexive**, because $x \neq x + 1$ and $x \neq x - 1$ for any integer x .

The relation R is **symmetric**, because if $x = y + 1$, then $y = x - 1$ and if $x = y - 1$, then $y = x + 1$.

The relation R is **not antisymmetric**, because if $x = 1$ and $y = 2$, then $(x, y) \in R$ and $(y, x) \in R$, while $x \neq y$.

The relation R is **not transitive**, because if $x = 1$, $y = 2$ and $z = 3$, then $(x, y) \in R$ and $(y, z) \in R$, while $(x, z) \notin R$.

(d)

$$R = \{(x, y) | x \equiv y \pmod{7}\}$$

The relation R is **reflexive**, because $x \equiv x \pmod{7}$ is true for any integer x .

The relation R is **symmetric**, because if $x \equiv y \pmod{7}$, then $y \equiv x \pmod{7}$.

The relation R is **not antisymmetric**, because if $x = 0$ and $y = 7$, then $x \equiv y \pmod{7}$ and $y \equiv x \pmod{7}$, while $x \neq y$.

The relation R is **transitive**, because if $x \equiv y \pmod{7}$ and $y \equiv z \pmod{7}$, then $x \equiv z \pmod{7}$.

(e)

$$R = \{(x, y) | x \text{ is a multiple of } y\}$$

The relation R is **reflexive**, because x is always a multiple of itself.

The relation R is **not symmetric**, because 2 is a multiple of 1, while 1 is not a multiple of 2.

The relation R is **not antisymmetric**, because -1 is a multiple of 1 and 1 is a multiple of -1 , while $1 \neq -1$.

The relation R is **transitive**, because if x is a multiple of y and if y is a multiple of z , then x also has to be a multiple of z .

(g)

$$R = \{(x, y) | x = y^2\}$$

The relation R is **not reflexive**, because (for example) $(2, 2)$ is not in R (since $2 \neq 2^2 = 4$).

The relation R is **not symmetric**, because if $x = 4$ and $y = 2$, then $4 = 2^2$ and $2 \neq 4^2$

The relation R is **antisymmetric**, because if $x = y^2$ and $y = x^2$, then $x = y^2 = (x^2)^2 = x^4$, which implies $x = y = 0$ or $x = y = 1$.

The relation R is **not transitive**, because if $x = 16$, $y = 4$ and $z = 2$, then $x = y^2$ and $y = z^2$, while $x \neq z^2$

(h)

$$R = \{(x, y) | x \geq y^2\}$$

The relation R is **not reflexive**, because (for example) $(2, 2)$ is not in R (since $2 < 2^2$).

The relation R is **not symmetric**, because if $x = 2$ and $y = 1$, then $2 \geq 1^2$ and $1 < 2^2$.

The relation R is **antisymmetric**, because if $x \geq y^2$ and if $y \geq x^2$, then $x \geq y^2 \geq (x^2)^2 = x^4$ and $x \geq x^4$ is only true if $x = 0$ or $x = 1$. When $x = 0$, then $y = 0$ also. When $x = 1$, then $1 \geq y^2 \geq 1$, thus $y^2 = 1$ and thus also $y = 1$ (as $y \geq x^2$).

The relation R is **transitive**, because if $x \geq y^2$ and if $y \geq z^2$, then $x \geq y^2 \geq y \geq z^2$.

Solution 4

DEFINITIONS

The **inverse relation** R^{-1} is the set $\{(b, a) | (a, b) \in R\}$

The **complementary relation** $\overline{R}$ is the set $\{(a, b) | (a, b) \notin R\}$

SOLUTION

$$R = \{(a, b) | a < b\}$$

(a) Use the definition of the inverse relation:

$$\begin{aligned} R^{-1} &= \{(b, a) | (a, b) \in R\} \\ &= \{(b, a) | a < b\} \\ &= \{(a, b) | b < a\} && \text{Interchange } a \text{ and } b \\ &= \{(a, b) | a > b\} \end{aligned}$$

(b) Use the definition of the complementary relation:

$$\begin{aligned} \overline{R} &= \{(a, b) | (a, b) \notin R\} \\ &= \{(a, b) | a \not< b\} \\ &= \{(a, b) | a \geq b\} \end{aligned}$$

Solution 5

DEFINITIONS

Union $A \cup B$: All elements that are either in A OR in B

Intersection $A \cap B$: All elements that are both in A AND in B .

Difference $A - B$: All elements in A that are NOT in B (complement of B with respect to A).

Symmetric difference $A \oplus B$: All elements in A or in B , but not in both.

SOLUTION

$$R_1 = \{(a, b) \in \mathbf{R}^2 | a > b\}$$

$$R_2 = \{(a, b) \in \mathbf{R}^2 | a \geq b\}$$

$$R_3 = \{(a, b) \in \mathbf{R}^2 | a < b\}$$

$$R_4 = \{(a, b) \in \mathbf{R}^2 | a \leq b\}$$

$$R_5 = \{(a, b) \in \mathbf{R}^2 | a = b\}$$

$$R_6 = \{(a, b) \in \mathbf{R}^2 | a \neq b\}$$

(a) The union of two relations contains all ordered pairs that are in either relation.

$$\begin{aligned} R_1 \cup R_3 &= \{(a, b) \in \mathbf{R}^2 | a > b \text{ or } a < b\} \\ &= \{(a, b) \in \mathbf{R}^2 | a \neq b\} \\ &= R_6 \end{aligned}$$

Because when $a < b$ or $b > a$, then we know that a and b cannot be the same (although we do not know anything more).

(b) The union of two relations contains all ordered pairs that are in either relation.

$$\begin{aligned} R_1 \cup R_5 &= \{(a, b) \in \mathbf{R}^2 | a > b \text{ or } a = b\} \\ &= \{(a, b) \in \mathbf{R}^2 | a \geq b\} \\ &= R_2 \end{aligned}$$

(c) The intersection of two relations contains all ordered pairs that are in both relations.

$$\begin{aligned} R_2 \cap R_4 &= \{(a, b) \in \mathbf{R}^2 | a \geq b \text{ and } a \leq b\} \\ &= \{(a, b) \in \mathbf{R}^2 | a = b\} \\ &= R_5 \end{aligned}$$

(d) The intersection of two relations contains all ordered pairs that are in both relations.

$$\begin{aligned} R_3 \cap R_5 &= \{(a, b) \in \mathbf{R}^2 | a < b \text{ and } a = b\} \\ &= \emptyset \end{aligned}$$

Since $a < b$ and $a = b$ can never be true at the same time, $R_3 \cap R_5$ cannot contain any ordered pairs.

(e) $R_1 - R_2$ contains all ordered pairs that are in the relation R_1 that do not occur in the relation R_2 .

$$\begin{aligned} R_1 - R_2 &= \{(a, b) \in \mathbf{R}^2 | a > b \text{ and } a \not\geq b\} \\ &= \{(a, b) \in \mathbf{R}^2 | a > b \text{ and } a < b\} \\ &= \emptyset \end{aligned}$$

Since $a > b$ and $a < b$ can never be true at the same time, $R_1 - R_2$ cannot contain any ordered pairs.

(f) $R_2 - R_1$ contains all ordered pairs that are in the relation R_2 that do not occur in the relation R_1 .

$$\begin{aligned} R_2 - R_1 &= \{(a, b) \in \mathbf{R}^2 | a \geq b \text{ and } a \not\leq b\} \\ &= \{(a, b) \in \mathbf{R}^2 | a \geq b \text{ and } a < b\} \\ &= \{(a, b) \in \mathbf{R}^2 | a = b\} \\ &= R_5 \end{aligned}$$

(g) $R_1 \oplus R_3$ contains all ordered pairs that are in the relation R_1 or are in the relation R_3 , but not in both.

$$\begin{aligned}
 R_1 \oplus R_3 &= \{(a, b) \in \mathbf{R}^2 | (a > b \text{ or } a < b) \text{ and not } (a > b \text{ and } a < b)\} \\
 &= \{(a, b) \in \mathbf{R}^2 | a \neq b \text{ and not } \mathbf{F}\} \\
 &= \{(a, b) \in \mathbf{R}^2 | a \neq b \text{ and } \neg \mathbf{F}\} \\
 &= \{(a, b) \in \mathbf{R}^2 | a \neq b \text{ and } \mathbf{T}\} && \neg \mathbf{F} \equiv \mathbf{T} \\
 &= \{(a, b) \in \mathbf{R}^2 | a \neq b\} && p \wedge \mathbf{T} \equiv p \\
 &= R_6
 \end{aligned}$$

Note that the statement $a > b$ and $a < b$ is always false $\mathbf{F}$ for all pairs of real numbers.

(h) $R_2 \oplus R_4$ contains all ordered pairs that are in the relation R_2 or are in the relation R_4 , but not in both.

$$\begin{aligned}
 R_2 \oplus R_4 &= \{(a, b) \in \mathbf{R}^2 | (a \geq b \text{ or } a \leq b) \text{ and not } (a \geq b \text{ and } a \leq b)\} \\
 &= \{(a, b) \in \mathbf{R}^2 | \mathbf{T} \text{ and not } (a = b)\} \\
 &= \{(a, b) \in \mathbf{R}^2 | \mathbf{T} \text{ and } \neg(a = b)\} \\
 &= \{(a, b) \in \mathbf{R}^2 | \mathbf{T} \text{ and } a \neq b\} \\
 &= \{(a, b) \in \mathbf{R}^2 | a \neq b\} && p \wedge \mathbf{T} \equiv p \\
 &= R_6
 \end{aligned}$$

Note that the statement $a \geq b$ or $a \leq b$ is always true $\mathbf{T}$ for all pairs of real numbers.

DEFINITIONS

The **composite** $S \circ R$ consists of all ordered pairs (a, c) for which there exists an element b such that $(a, b) \in R$ and $(b, c) \in S$

SOLUTION

$$R_1 = \{(a, b) \in \mathbf{R}^2 | a > b\}$$

$$R_2 = \{(a, b) \in \mathbf{R}^2 | a \geq b\}$$

$$R_3 = \{(a, b) \in \mathbf{R}^2 | a < b\}$$

$$R_4 = \{(a, b) \in \mathbf{R}^2 | a \leq b\}$$

$$R_5 = \{(a, b) \in \mathbf{R}^2 | a = b\}$$

$$R_6 = \{(a, b) \in \mathbf{R}^2 | a \neq b\}$$

(a) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned} R_1 \circ R_1 &= \{(a, c) \in \mathbf{R}^2 | \exists b \in \mathbf{R} : (a, b) \in R_1 \text{ and } (b, c) \in R_1\} \\ &= \{(a, c) \in \mathbf{R}^2 | \exists b \in \mathbf{R} : a > b \text{ and } b > c\} \\ &= \{(a, c) \in \mathbf{R}^2 | \exists b \in \mathbf{R} : a > c\} \\ &= \{(a, c) \in \mathbf{R}^2 | a > c\} \\ &= \{(a, b) \in \mathbf{R}^2 | a > b\} \\ &= R_1 \end{aligned}$$

$a > c$ is independent of b

Let us rename c as b

(b) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned} R_1 \circ R_2 &= \{(a, c) \in \mathbf{R}^2 | \exists b \in \mathbf{R} : (a, b) \in R_2 \text{ and } (b, c) \in R_1\} \\ &= \{(a, c) \in \mathbf{R}^2 | \exists b \in \mathbf{R} : a \geq b \text{ and } b > c\} \\ &= \{(a, c) \in \mathbf{R}^2 | \exists b \in \mathbf{R} : a > c\} \\ &= \{(a, c) \in \mathbf{R}^2 | a > c\} \\ &= \{(a, b) \in \mathbf{R}^2 | a > b\} \\ &= R_1 \end{aligned}$$

$a > c$ is independent of b

Let us rename c as b

(c) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned} R_1 \circ R_3 &= \{(a, c) \in \mathbf{R}^2 | \exists b \in \mathbf{R} : (a, b) \in R_3 \text{ and } (b, c) \in R_1\} \\ &= \{(a, c) \in \mathbf{R}^2 | \exists b \in \mathbf{R} : a < b \text{ and } b > c\} \\ &= \{(a, c) \in \mathbf{R}^2 | \mathbf{T}\} \\ &= \{(a, c) \in \mathbf{R}^2\} \\ &= \mathbf{R}^2 \end{aligned}$$

$\exists b \in \mathbf{R} : a < b \text{ and } b > c$ is always true

$\exists b \in \mathbf{R} : a < b \text{ and } b > c$ is always true, because we can always chose $b = \max(a, c) + 1$.

(d) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned}
 R_1 \circ R_4 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_4 \text{ and } (b, c) \in R_1\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a \leq b \text{ and } b > c\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \mathbf{T}\} && \exists b \in \mathbf{R} : a \leq b \text{ and } b > c \text{ is always true} \\
 &= \{(a, c) \in \mathbf{R}^2\} \\
 &= \mathbf{R}^2
 \end{aligned}$$

$\exists b \in \mathbf{R} : a \leq b \text{ and } b > c$ is always true, because we can always chose $b = \max(a, c) + 1$.

(e) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned}
 R_1 \circ R_5 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_5 \text{ and } (b, c) \in R_1\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a = b \text{ and } b > c\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a > c\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid a > c\} && a > c \text{ is independent of } b \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a > b\} && \text{Let us rename } c \text{ as } b \\
 &= R_1
 \end{aligned}$$

(f) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned}
 R_1 \circ R_6 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_6 \text{ and } (b, c) \in R_1\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a \neq b \text{ and } b > c\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \mathbf{T}\} && \exists b \in \mathbf{R} : a \neq b \text{ and } b > c \text{ is always true} \\
 &= \{(a, c) \in \mathbf{R}^2\} \\
 &= \mathbf{R}^2
 \end{aligned}$$

$\exists b \in \mathbf{R} : a \neq b \text{ and } b > c$ is always true, because we can chose $b = c + 1$ when $c + 1 \neq a$ and $b = c + 2$ when $c + 1 = a$.

(g) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned}
 R_2 \circ R_3 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_3 \text{ and } (b, c) \in R_2\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a < b \text{ and } b \geq c\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \mathbf{T}\} && \exists b \in \mathbf{R} : a < b \text{ and } b \geq c \text{ is always true} \\
 &= \{(a, c) \in \mathbf{R}^2\} \\
 &= \mathbf{R}^2
 \end{aligned}$$

$\exists b \in \mathbf{R} : a < b \text{ and } b \geq c$ is always true, because we can chose $b = \max(a, c) + 1$.

(h) For every pair $(a, b) \in R$ and $(b, c) \in S$, the corresponding element (a, c) belongs to the composition.

$$\begin{aligned}
 R_3 \circ R_3 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : (a, b) \in R_3 \text{ and } (b, c) \in R_3\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a < b \text{ and } b < c\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid \exists b \in \mathbf{R} : a < c\} \\
 &= \{(a, c) \in \mathbf{R}^2 \mid a < c\} && a < c \text{ is independent of } b \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a < b\} && \text{Let us rename } c \text{ as } b \\
 &= R_3
 \end{aligned}$$

Solution 6

SOLUTION

$$R_1 = \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b\}$$

$$R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \text{ is a multiple of } b\}$$

(a) The union of two relations contains all ordered pairs that are in either relation.

$$\begin{aligned}
 R_1 \cup R_2 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ or } a \text{ is a multiple of } b\} \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ or } b \text{ divides } a\}
 \end{aligned}$$

Note: a is a multiple of b if and only if b divides a .

(b) The intersection of two relations contains all ordered pairs that are in both relations.

$$\begin{aligned}
 R_1 \cap R_2 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ and } a \text{ is a multiple of } b\} \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ and } b \text{ divides } a\} \\
 &= \{(a, b) \in \mathbf{R}^2 \mid a = \pm b \text{ and } a \neq 0\}
 \end{aligned}$$

Note: You cannot divide by 0.

(c) $R_1 - R_2$ contains all ordered pairs that are in the relation R_1 that do not occur in the relation R_2 .

$$\begin{aligned} R_1 - R_2 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ and } a \text{ is not a multiple of } b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ and } b \text{ does not divide } a\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ and } a \neq \pm b\} \end{aligned}$$

(d) $R_2 - R_1$ contains all ordered pairs that are in the relation R_2 that do not occur in the relation R_1 .

$$\begin{aligned} R_2 - R_1 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ is a multiple of } b \text{ and } a \text{ does not divide } b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid b \text{ divides } a \text{ and } a \text{ does not divide } b\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid b \text{ divides } a \text{ and } a \neq \pm b\} \end{aligned}$$

(e) $R_1 \oplus R_2$ contains all ordered pairs that are in the relation R_1 or R_2 , but not in both.

$$\begin{aligned} R_1 \oplus R_2 &= \{(a, b) \in \mathbf{R}^2 \mid a \text{ divides } b \text{ or } b \text{ is a multiple of } a, \text{ but not both}\} \\ &= \{(a, b) \in \mathbf{R}^2 \mid (a \text{ divides } b \text{ or } b \text{ divides } a) \text{ and } a \neq \pm b\} \end{aligned}$$

Solution 7

DEFINITIONS

A relation R on a set A is **reflexive** if $(a, a) \in R$ for every element $a \in A$.

A relation R on a set A is **transitive** if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

The **composite** $S \circ R$ consists of all ordered pairs (a, c) for which there exists an element b such that $(a, b) \in R$ and $(b, c) \in S$.

$$R^2 = (1,2)(1,3)(1,4)(1,1)(1,5)(2,1)(2,4)(2,5)(2,2)(3,2)(3,3)(3,4)(3,5)(3,1)(4,3)(4,4)(4,2)(4,1)(5,1)(5,2)(5,3)(5,4)(5,5)$$

$$R^3 = (1,2)(1,3)(1,4)(1,1)(1,5)(2,2)(2,3)(2,4)(2,5)(2,1)(3,2)(3,3)(3,4)(3,1)(3,5)(4,1)(4,4)(4,5)(4,2)(4,3)(5,2)(5,3)(5,4)(5,1)(5,5)$$

$$R^4 = (1,1)(1,2)(1,3)(1,4)(1,5)(2,1)(2,2)(2,3)(2,4)(2,5)(3,2)(3,3)(3,4)(3,5)(3,1)(4,3)(4,4)(4,2)(4,1)(4,5)(5,1)(5,2)(5,3)(5,4)(5,5)$$

$$R^5 = (1,2)(1,1)(1,3)(1,5)(1,4)(2,1)(2,2)(2,3)(2,4)(2,5)(3,2)(3,3)(3,4)(3,5)(3,1)(4,3)(4,4)(4,2)(4,1)(4,5)(5,1)(5,2)(5,3)(5,4)(5,5)$$

Solution 8

DEFINITIONS

The **inverse relation** R^{-1} is the set $\{(b, a) | (a, b) \in R\}$

The **complementary relation** $\overline{R}$ is the set $\{(a, b) | (a, b) \notin R\}$

The **composite** $S \circ R$ consists of all ordered pairs (a, c) for which there exists an element b such that $(a, b) \in R$ and $(b, c) \in S$

A relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

SOLUTION

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) The matrix corresponding to the inverse relation R^{-1} is the **transposed** of the matrix representing R :

$$\begin{aligned} \mathbf{M}_{R^{-1}} &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \mathbf{M}_R \end{aligned}$$

(b) The matrix corresponding to the complementary relation $\overline{R}$ **changes every zero to a 1 and changes every one to a 0** in the matrix representing R :

$$\mathbf{M}_{\overline{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) The matrix corresponding to the composite of two matrices is the Boolean product of two matrices.

$$\begin{aligned}
 \mathbf{M}_{R^2} &= \mathbf{M}_R \odot \mathbf{M}_R \\
 &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} (0 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1) \\ (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 1) \end{bmatrix} \\
 &= \begin{bmatrix} 0 \vee 1 \vee 1 & 0 \vee 1 \vee 0 & 0 \vee 0 \vee 1 \\ 0 \vee 1 \vee 0 & 1 \vee 1 \vee 0 & 1 \vee 0 \vee 0 \\ 0 \vee 0 \vee 1 & 1 \vee 0 \vee 0 & 1 \vee 0 \vee 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
 \end{aligned}$$

Solution 9
SOLUTION

$$\begin{aligned}
 \mathbf{M}_{R_1} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
 \mathbf{M}_{R_2} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
 \end{aligned}$$

(a) The matrix corresponding to the union of two relations is the **join** of the matrices representing each of the relations:

$$\begin{aligned}
 \mathbf{M}_{R_1 \cup R_2} &= \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \\
 &= \begin{bmatrix} 0 \vee 0 & 1 \vee 1 & 0 \vee 0 \\ 1 \vee 0 & 1 \vee 1 & 1 \vee 1 \\ 1 \vee 1 & 0 \vee 1 & 0 \vee 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
 \end{aligned}$$

(b) The matrix corresponding to the intersection of two relations is the **meet** of the matrices representing each of the relations:

$$\begin{aligned}
 \mathbf{M}_{R_1 \cap R_2} &= \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} \\
 &= \begin{bmatrix} 0 \wedge 0 & 1 \wedge 1 & 0 \wedge 0 \\ 1 \wedge 0 & 1 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 1 & 0 \wedge 1 & 0 \wedge 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

(c) The matrix corresponding to the composite of two matrices is the Boolean product of two matrices.

$$\begin{aligned}
 \mathbf{M}_{R_2 \circ R_1} &= \mathbf{M}_{R_1} \odot \mathbf{M}_{R_2} \\
 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} (0 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) \\ (1 \wedge 0) \vee (1 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) \\ (1 \wedge 0) \vee (0 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 1) \end{bmatrix} \\
 &= \begin{bmatrix} 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 1 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}
 \end{aligned}$$

(d) The matrix corresponding to the composite of two matrices is the Boolean product of two matrices.

$$\begin{aligned}
 \mathbf{M}_{R_1 \circ R_1} &= \mathbf{M}_{R_1} \odot \mathbf{M}_{R_1} \\
 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) & (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 0) \\ (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 0) \\ (1 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 0) \end{bmatrix} \\
 &= \begin{bmatrix} 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 1 \vee 1 & 1 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}
 \end{aligned}$$

(e) The matrix corresponding to the union of two relations is the **join** of the matrices representing each of the relations:

$$\begin{aligned}\mathbf{M}_{R_1 \oplus R_2} &= \mathbf{M}_{R_1} \oplus \mathbf{M}_{R_2} \\ &= \begin{bmatrix} 0 \oplus 0 & 1 \oplus 1 & 0 \oplus 0 \\ 1 \oplus 0 & 1 \oplus 1 & 1 \oplus 1 \\ 1 \oplus 1 & 0 \oplus 1 & 0 \oplus 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\end{aligned}$$

Solution 10

SOLUTION

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(a) The matrix corresponding to the composite of two relations is the Boolean product of two corresponding matrices.

$$\begin{aligned}\mathbf{M}_{R^2} &= \mathbf{M}_R \odot \mathbf{M}_R \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (0 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 0) \\ (0 \wedge 0) \vee (0 \wedge 0) \vee (1 \wedge 1) & (0 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 1) & (0 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (1 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 0) \end{bmatrix} \\ &= \begin{bmatrix} 0 \vee 0 \vee 0 & 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 1 \vee 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\end{aligned}$$

(b) The matrix corresponding to the composite of two relations is the Boolean product of two corresponding matrices.

$$\begin{aligned}
\mathbf{M}_{R^3} &= \mathbf{M}_{R^2} \odot \mathbf{M}_R \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} (0 \wedge 0) \vee (0 \wedge 0) \vee (1 \wedge 1) & (0 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 1) & (0 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (1 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 0) \\ (0 \wedge 0) \vee (1 \wedge 0) \vee (1 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 0) \end{bmatrix} \\
&= \begin{bmatrix} 0 \vee 0 \vee 1 & 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 0 \vee 0 \vee 1 & 0 \vee 1 \vee 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}$$

(c) The matrix corresponding to the composite of two relations is the Boolean product of two corresponding matrices.

$$\begin{aligned}
\mathbf{M}_{R^3} &= \mathbf{M}_{R^2} \odot \mathbf{M}_R \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} (1 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 0) \\ (0 \wedge 0) \vee (1 \wedge 0) \vee (1 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 0) \\ (1 \wedge 0) \vee (1 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 0) \end{bmatrix} \\
&= \begin{bmatrix} 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 0 \vee 0 \vee 1 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 1 \vee 0 \vee 1 & 0 \vee 1 \vee 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}$$

Solution 11

- a) $R = \{(x, y) : x - y \in \mathbb{Z}\} \Rightarrow [1]_R = \{y : (1, y) \in R\} = \{y : 1 - y \in \mathbb{Z}\} = \mathbb{Z}.$
b) $R = \{(x, y) : x - y \in \mathbb{Z}\} \Rightarrow [1/2]_R = \{y : (\frac{1}{2}, y) \in R\} = \{y : \frac{1}{2} - y \in \mathbb{Z}\} = \{(2n+1)/2 : n \in \mathbb{Z}\}.$

